National Exams December 2019

16-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. This is a Closed Book exam.
   Candidates may use one of two calculators, the Casio or Sharp approved models. They are also entitled to one aid sheet with tables & formulas written both sides. No textbook excerpts or examples solved.

3. FIVE (5) questions constitute a complete exam.
   Clearly indicate your choice of any five of the six questions given otherwise the first five answers found will be considered your pick.

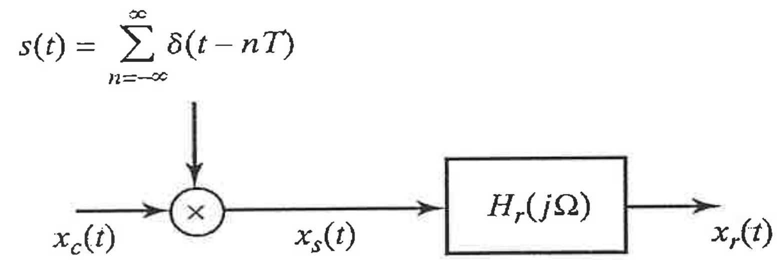
4. All questions are worth 12 points.
   See below for a detailed breakdown of the marking.

Marking Scheme

1. (a) 3, (b) 3, (c) 3, (d) 3, total = 12
2. (a) 3, (b) 3, (c) 3, (d) 3, total = 12
3. (a) 3, (b) 6, (c) 3, total = 12
4. (a) 3, (b) 4, (c) 5, total = 12
5. (a) 2, (b) 2, (c) 2, (d) 2, (e) 2, (f) 2, total = 12
6. (a) 4, (b) 4, (c) 4, total = 12

The number beside each part above indicates the points that part is worth

1. Consider the representation of the process of sampling followed by reconstruction shown in the figure below.



Assume that the input signal is

$$x_c(t) = 2cos(100\pi t - \pi/4) + cos(300\pi t + \pi/3), \qquad -\infty < t < \infty.$$

The frequency response of the reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \pi/T \\ 0, & |\Omega| > \pi/T \end{cases}$$

(a) Determine the continuous-time Fourier transform $X_c(j\Omega)$. Plot it as a function of Ω.

(b) Assume that $f_s = 1/T = 500\ samples/sec$ and plot the Fourier transform $X_s(j\Omega)$ as a function of Ω for $-2\pi/T \le \Omega \le 2\pi/T$. What is the output $x_r(t)$ in this case? (You should be able to give the exact analytical expression for $x_r(t)$).

(c) Now, assume that $f_s = 1/T = 250\ samples/sec.$ Repeat part (b) for this condition.

(d) Is it possible to choose the sampling rate so that

$$x_r(t) = A + 2cos(100\pi t - \pi/4)$$

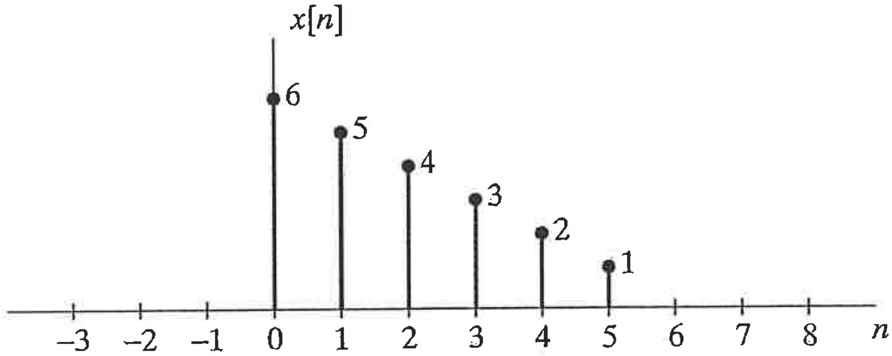
where A is a constant?

If so, what is the sampling rate $f_s = 1/T$, and what is the numerical value of A?

2.- Consider the six-point sequence

$$x[n] = 6\delta[n] + 5\delta[n-1] + 4\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$

shown in the figure below



(a) Determine $X[k]$, the six-point DFT of $x[n]$.
Express your answer in terms of $W_6 = e^{-j2\pi/6}$.

(b) Plot the sequence $w[n]$, $n = 0, 1, \ldots, 5$, that is obtained by computing the inverse six-point DFT of $W[k] = W_6^{-2k} X[k]$.

(c) Use any convenient method to find the six-point circular convolution of $x[n]$ with the sequence $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$. Sketch the result.

(d) If we convolve the given $x[n]$ with the given $h[n]$ by using an N-point circular convolution, what value should be chosen for N such that the result of the circular convolution is identical to the result of the linear convolution? Explain.

3.- The system function of a causal LTI system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4} z^{-1}}$$

The input to this system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$$

(a) Find the impulse response of the system, $h[n]$.

(b) Find the output $y[n]$.

(c) Is the system stable? Justify.

4.- A causal LTI system has system function given by the following expression:

$$H(z) = \frac{1}{1 - z^{-1}} + \frac{1 - z^{-1}}{1 - z^{-1} + 0.8z^{-2}}.$$

(a) Is this system stable? Explain briefly.

(b) Draw the signal flow graph of a parallel form implementation of this system. Use a direct form II implementation for the 2^{nd}-order subsystem.

(c) Draw the signal flow graph of a cascade form implementation of this system as a cascade of a 1^{st}-order system and a 2^{nd}-order system. Use a transposed direct form II implementation for the 2^{nd}-order system.

5.- We wish to use the Kaiser window method to design a discrete-time linear phase FIR filter that meets the following specifications:

$$0.95 \le |H(e^{j\omega})| \le 1.05, \qquad 0 \le |\omega| \le 0.2\pi,$$

$$|H(e^{j\omega})| \le 0.06, \qquad 0.3\pi \le |\omega| \le 0.475\pi,$$

$$1.95 \le |H(e^{j\omega})| \le 2.05, \qquad 0.525\pi \le |\omega| \le \pi.$$

Kaiser formulas:

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50, \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases} \qquad where\ A = -20\log_{10}\delta,$$

$$M = \frac{A - 8}{2.285\Delta\omega} \qquad where\ \Delta\omega\ is\ the\ transition\ band\ width\ in\ the\ design\ specifications.$$

(a) What is the highest value of the tolerance δ that can be used to meet the specifications? Explain.

(b) What is the corresponding value of β?

(c) What is the highest value of $\Delta\omega$ to be used to meet the specifications? Explain.

(d) Find the length of the shortest impulse response $h[n]$, *i.e.* least number of filter coefficients, that will satisfy the specifications above. Explain.

(e) What is the delay introduced by the resulting filter in number of samples?

(f) Provide the ideal impulse response $h_d[n]$ to which the Kaiser window should be applied to obtain $h[n]$.

6.- (a) During the design of an IIR digital filter using the bilinear transformation, the s-plane poles obtained in the design of the Butterworth analogue filter prototype based on $|H(j\Omega)|^2$ are:

$p_1 = 0.71e^{j\pi/12}$	$p_5 = 0.71e^{j9\pi/12}$	$p_9 = 0.71e^{j17\pi/12}$
$p_2 = 0.71e^{j3\pi/12}$	$p_6 = 0.71e^{j11\pi/12}$	$p_{10} = 0.71e^{j19\pi/12}$
$p_3 = 0.71e^{j5\pi/12}$	$p_7 = 0.71e^{j13\pi/12}$	$p_{11} = 0.71e^{j21\pi/12}$
$p_4 = 0.71e^{j7\pi/12}$	$p_8 = 0.71e^{j15\pi/12}$	$p_{12} = 0.71e^{j23\pi/12}$

(i) Which poles would you choose to form the analogue filter response $H(s)$? Explain why.

(ii) How would you then find the digital filter response $H(z)$ from $H(s)$?

(b) You are to design a lowpass IIR digital filter using the impulse invariance transformation. During the design of the Butterworth analogue filter prototype you solve a system of two equations determined by the desired frequency response for the pass-band corner frequency and the stop-band corner frequency.

The values resulting for the Butterworth analogue filter parameters are:

$\Omega_c = 0.815,$ and $N = 5.305$

(i) What should your next step be?

(ii) If you were to exactly meet the desired frequency response specifications for the passband corner frequency or the stopband corner frequency, which one of the two would you choose to meet? Explain why.

(c) There are four types of linear-phase FIR digital filters. Not all of them are able to accommodate all types of frequency selective filters; lowpass, highpass, bandpass and bandstop.

In the table below mark with an X those implementations that are not possible to occur and justify why not based on the z-plane forced zero locations of $H(z)$ for each filter type.

FIR Filter	Lowpass	Highpass	Bandpass	Bandstop
Type I				
Type II				
Type III				
Type IV				

Additional Information

(Not all of this information is necessarily required today!)

DTFT Synthesis Equation	DTFT Analysis Equation
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
Parseval's Theorem	**N-point DFT**
$E = \sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$	$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \quad W_N = e^{-j\frac{2\pi}{N}}$
Z-transform of a sequence $x[n]$	**Sinusoidal response of LTI systems, real h[n]**
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$y[n] = \lvert H(e^{j\omega_0})\rvert \cos\left(\omega_0 n + \measuredangle H(e^{j\omega_0})\right)$

SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Additional Information (cont'd)

Geometric Sum	Geometric Series
$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$	$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad \lvert q\rvert < 1$

Properties of the Discrete Fourier Transform

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km}X[k]$
6.	$W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{I}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Initial Value Theorem:

If $x[n]$ is a causal sequence, *i.e.* $x[n] = 0, \forall n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$